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FREE WAVES IN PERIODICALLY DISORDERED SYSTEMS: NATURAL AND BOUNDING FREQUENCIES OF UNSYMMETRIC SYSTEMS AND NORMAL MODE LOCALIZATION

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A general method has been developed and applied to study the effect of different amounts of disorders on free flexural wave motion in undamped beam-type systems consisting of finite multi-span repeating units that are disordered identically due to (i) unequal support spacings and (ii) the presence of point masses and point springs at some of the supports. The frequency primary propagation zones (pass bands) of periodic systems are, in general, divided into as many intermediate propagation zones as the number of beam elements of the disordered repeating units. In the case of systems with symmetrically disordered repeating units, the frequency propagation zones are always bound by the frequencies that are identified with natural frequencies of the repeating unit with the extreme ends simply supported or clamped. This is not the case when the repeating units are disordered unsymmetrically. Natural frequencies of unsymmetric multi-span beams normally lie inside the attenuation zones (stop bands) and the conditions under which they can lie at the bounds and even inside the propagation zones have been identified. The presence of disorders normally interferes with free wave motion and narrows down the effective frequency bands of free wave propagation, but this is not always true. Conditions have been identified under which some specific beam length disorders do not interfere with the free wave propagation in certain frequency bands and can even broaden such bands. It is also explained how the transmission of waves (and vibrations) can be controlled by introducing appropriate disorders. Confinement of free waves corresponding to normal modes of disordered systems has been discussed in terms of the attenuation constant of free wave motion. It is argued on the basis of present and earlier studies on periodically disordered systems, that the normal modes only of the unsymmetrically disordered systems that lie in attenuation zones can become localized (even when the system elements are not weakly coupled). The influence of variation of coupling, disorder, damping and that of the frequency on wave confinement is discussed and interpreted qualitatively in terms of the attenuation (decay) constant of periodically disordered systems. The effect of these parameters is found to compare well in general with their well-known effects on normal mode localization in beam-type structures.

1. INTRODUCTION

Free wave motion in infinite periodic systems has long been studied in the field of solid state physics and engineering structures [1–5]. Beam-type systems, which are idealized models of aeronautical and marine structures, have been considered to be ideally periodic such that all the periodic elements (or the repeating units) are identical and they are attached to the adjoining elements or units in an identical manner. Free wave motion in such systems is governed by a propagation constant, a complex quantity. It is purely

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in the frequency propagation zones (pass bands) in which the waves travel freely along the system without spatial decay. Free waves attenuate or decay exponentially as they travel in the remaining alternate frequency bands known as attenuation zones (stop bands).

In reality, engineering structures are not ideally periodic and they are disordered in the sense that the periodic units are not exactly identical. The parameter values of the so-called "periodic elements" can at best be specified within a small predetermined range due to, for example, our inability to manufacture, measure and calculate them exactly [6–8]. It is known that the random imperfection in spatial periodicity results in variability in normal modes. It has also been established that the frequency response function could be considerably greater than the value computed without taking into account the disorder in span lengths, especially near the frequencies of resonance. In these studies, the beam systems were considered to be resting on simple supports which offered no torsional stiffness and were thus strongly coupled.

Localized vibrations in a linear diatomic chain having a defect (disorder) at the centre of the chain was reported by Bjork [9]. Beam-type infinite periodic structures that are disordered only locally over a single element [10] and a group of elements [11] have also been studied. Beam length disorders and point mass/spring disorders attenuate transmission of flexural waves in the frequency propagation zones (PZs) of the periodic system. More recently, Langley [12] has discussed wave transmission through systems in which an in-line finite system is assumed to be embedded into an infinite wave guide. The general conclusions drawn are somewhat similar to those of reference [11].

The only work on engineering structures in acoustical terms reported on transmission and attenuation of waves across a disordered system, after the work presented in references [10, 11], was by Glushkov and Kuz'michev in 1980 [13] and Wanzel in 1982 [14], before Hodges [15] and Hodges and Woodhouse [16], who were the first to study the attenuation caused by the disordered systems in terms of normal mode localization (or confinement), as earlier reported by Anderson in 1958 [17] in solid state physics. Hodges and Woodhouse provided an excellent explanation of the normal mode localization on the basis of theoretical and experimental studies, and called it "Anderson Localization". The models that they used consisted of weakly coupled series of pendula and masses attached to stretched wires. Comparatively more recently, localization has also been studied extensively in nearly periodic (randomly disordered or mistuned) mono-coupled beam-type structures [18–24]. The effect of span length deviation in weakly coupled beams (beams on simple supports offering high torsional stiffness) results in strong localization that is manifested by a large deflection close to excitation, and it decays to negligibly small values only a few spans away. Thus, for practical purposes, localization is somewhat similar to damping, as it obstructs the flow of energy and will thus remain confined near to the source of excitation. For damping, the energy is dissipated as it propagates. Although the localization is beneficial in obstructing the transmission of vibration, it can also prove destructive when the localized amplitudes are large and induce high local stresses. It is known that even slight disorder [21] can have a drastic effect on the dynamics of a nominally periodic system, provided that the coupling between the subsystems is sufficiently weak. In the present study, the phenomenon of localization in unsymmetrically disordered systems has been discussed in the light of the attenuation constant, which is not zero over the entire primary propagation zones of periodically disordered systems.

As the disorders in periodically disordered systems lie only within the repeating units, they can still be studied as periodic systems. Free wave motion in such systems has been studied earlier in the field of solid state physics [25] and also in beam-type systems [26, 27]. It is known that the primary propagation zones (PPZs) of such systems are divided into as many intermediate propagation zones (IPZs) as the number of disordered beam elements, in general. This is due to the appearance of intermediate attenuation zones (IAZs) within the PPZs. Unlike the PPZs, the primary attenuation zones (PAZs) of the disordered system do not split up but they normally become narrow. Thus the PPZs and PAZs of the disordered systems are, in general, broader and narrower than the PZs and AZs of the corresponding periodic systems, respectively. The additional advantage of studying the finite system as one of the repeating units lies in the fact that one can have better insight into the pass and stop bands and attenuation of free waves. It is known [27–29], and also shown in the present study, that the bounding frequencies of propagation zones are always identified with the natural frequencies of the finite system when disordered symmetrically. Also when the finite repeating unit consisting of N elements is disordered unsymmetrically, all the resonance frequencies lie inside the primary and intermediate attenuation zones [26, 27]; although for a periodic finite system consisting of N unsymmetric elements, $N-1$ of the natural frequencies lie inside the propagation zone and one of the frequencies lies inside the attenuation zone [28], with extreme ends either simply supported or clamped. It however remains to explore the conditions under which unsymmetrically disordered system can behave like a symmetric/period system, such that their natural frequencies can lie at the bounds or even inside the propagation zones. Furthermore, it is intended to study the effect of introducing different amounts of disorders in the repeating unit that can be used to avoid transmission of waves or vibrations over a specific frequency range.

General expressions for the natural frequencies and propagation constants that govern free wave motion are first set up in terms of the end receptances [30] of finite repeating units consisting of N different mono-coupled elements. The general expressions are then adapted to study the free wave motion in beams on simple supports. The repeating units considered are two-span beams with and without a point disorder (rotary mass or torsional spring) attached at their intermediate supports. The effect of point disorder on free wave propagation is studied. Variation of the bounding frequencies (and hence the pass and stop band widths) with the location of the intermediate support (prop) of the two-span beam is studied and compared to the corresponding values of the natural frequencies of the beam [31] with extreme ends simply supported (SS ends) and clamped (CC ends). The way in which deliberate disorders can be introduced in the repeating units to control free wave propagation is explained.

Finally, the confinement of free waves and normal modes is discussed in terms of the attenuation constant of periodically disordered systems consisting of unsymmetrically disordered finite repeating units. The present study is also helpful in understanding the phenomenon of normal mode localization in randomly disordered systems, because such systems are always expected to be unsymmetric. The normal modes can be strongly localized when the deliberate (deterministic) disorder is large and/or when the normal modes lie in the IAZs that belong to the higher PZs (pass bands) of the ordered systems. This is because the IAZs grow in size in the higher pass bands [26]. Individual and combined effects of coupling, beam length deviation from perfect periodicity (disorder), damping and frequency on the confinement of free waves and normal modes is discussed on the basis of the present study and the existing literature on periodically disordered systems [26, 27]. The literature on periodic structures, thought to be misleading in studying normal mode localization [15], is also argued to be relevant (at least qualitatively at this stage) when unsymmetrically disordered finite systems are considered to be the repeating units of the periodic structures. However, the quantitative comparison remains to be carried out and should form part of a future study.

2. GENERAL THEORY

Consider an infinite periodic system consisting of a large number of identical finite repeating units (FRU). In Figure 1(a) is shown a series of such units connected end to end in an identical manner, and in Figure 1(b) is shown a block diagram of the chain of N different elements $E_1, E_2, E_3, \ldots, F_N$ of the finite repeating unit connected to adjacent repeating units at co-ordinates A and B. To keep the analysis simple, all of the system elements are assumed to be coupled through single co-ordinates.

When the system vibrates freely, the forces acting on various elements will only be at the ends of the elements, where they are connected to the adjacent vibrating elements through common coupling co-ordinates. The forces $F_A e^{j\omega t}$ and $F_B e^{j\omega t}$ and displacements $q_A e^{j\omega t}$ and $q_B e^{i\omega t}$ that exist at the end co-ordinates A and B of the repeating unit, while the infinite system executes free motion, are also shown in Figure 1(b).

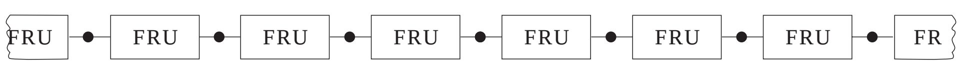
Since the infinite system is periodic, and the disorders lie only within the identical repeating units, due to their elements being different from one another, the displacements and forces at the periodic co-ordinates A and B are related through the end receptances of the finite system [30]. The displacements at the end co-ordinates of the repeating unit can be written in terms of its end receptances and the forces that exist there, such that

$$q_A = \alpha_{CAA} F_A - \beta_{CAB} F_B, \qquad q_B = -\alpha_{CBB} F_B + \beta_{CBA} F_A. \tag{1}$$

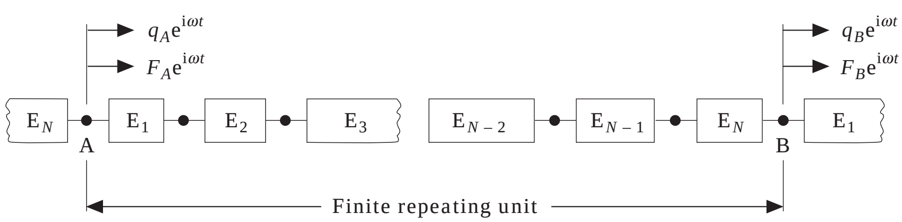
α_{CAA} and α_{CBB} are the direct end receptances and β_{CAB} and β_{CBA} are the transfer end receptances of the finite repeating unit. These are frequency dependent functions and are always real for undamped system elements. End receptances of finite systems are derived in reference [31]; these are listed in Appendix A for an array of N general elements. When the system elements are linear, one has $\beta_{CAB} = \beta_{CBA} = \beta_c$. This reduces the number of receptances to three, and equation (1) can now be written as

$$\begin{Bmatrix} q_A \\ q_B \end{Bmatrix} = \begin{bmatrix} \alpha_{CAA} & -\beta_c \\ \beta_c & -\alpha_{CBB} \end{bmatrix} \begin{Bmatrix} F_A \\ F_B \end{Bmatrix}. \tag{2}$$

When the finite repeating unit is disconnected from its neighbours, its extreme ends are free. The frequencies at which the receptances attain infinite values are the natural frequencies of the finite system with its extreme ends free. If the extreme ends of the finite



(a)



(b)

Figure 1. Block diagrams of (a) an infinite periodic system consisting of large number of finite repeating units (FRU) and (b) a chain of N individual elements of the repeating unit with the forces and displacements shown at the co-ordinates where it is coupled to the adjacent repeating units.

system are fixes, its end displacements q_A and q_B will always be zero. The eigenvalues for which these conditions are satisfied are given by

$$\begin{vmatrix} \alpha_{CAA} & \beta_c \\ -\beta_C & -\alpha_{CBB} \end{vmatrix} = 0 \tag{2a}$$

which gives the natural frequencies of the system with extreme ends fixed. The fixed end forces F_A and F_B, which are the eigenvectors, can be found from equation (2), corresponding to the eigenvalues found from equation (2a) with $q_A = q_B = 0$.

When free wave motion occurs in the infinite system the displacements and the forces at any two adjacent coupling co-ordinates of the repeating units are related through the progagation constant μ_D [4, 26] by

$$\begin{Bmatrix} q_B \\ F_B \end{Bmatrix} = \begin{Bmatrix} q_A \\ F_A \end{Bmatrix} e^{\mu_D}. \tag{3}$$

Substitution of q_B and F_B in terms of q_A and F_A into equations (2) and elimination of q_A and F_B yields, on rearrangement,

$$\cosh(\mu_D) = (\alpha_{CAA} + \alpha_{CBB})/2\beta_c. \tag{4}$$

When the system elements are symmetric, the direct end receptances of the repeating unit are equal. Thus $\alpha_{CAA} = \alpha_{CBB} = \alpha_c$ and one has $\cosh(\mu_D) = \alpha_c/\beta_c$. The propagation constant (μ, now μ_D) is an even function and has already been explained [4, 26]. It is a complex quantity in general, with $\mu_D = (\mu_D + i\mu_{Di})$. It is real only over certain frequency bands in which its imaginary part μ_{Di} is either zero or π. Free waves attenuate in these bands, which are known as attenuation zones (AZs). Free waves propagate in the systems only when μ_D is purely imaginary. The frequency bands in which free waves propagate are known as the propagation zones (PZs). These are bound by the frequencies at which $\cosh \mu_D = \pm 1$.

When the system elements are symmetric, one has $\cosh(\mu_D) = \alpha_c/\beta_c = \pm 1$. The frequencies at which $\alpha_c = \pm\beta_c$ [28] are also the bounding frequencies of the propagation zones, and are identified with the natural frequencies with extreme ends either free or fixed [26]. This is just the same as in the case of a periodic system with a symmetric single-element repeating unit [28]. For the periodic system with an unsymmetric single-element repeating unit, the natural frequencies of the element with both of its ends either free or fixed lie in attenuation zones and do not coincide with the bounding frequencies [28]. This is also true in general for periodic systems with an unsymmetric multiple-element repeating unit [26, 27], but it remains to study the conditions under which it is not so. For this purpose, free wave motion in some simple beam-type systems will be considered in a later section.

3. APPLICATIONS TO BEAM-TYPE SYSTEMS

The theory developed in the preceding section can be applied to study the free wave motion in any type of system the elements of which are mono-coupled and can be characterized by their end receptances. The theory developed will, however, be applied here only to study the natural frequencies and free wave motion in multi-supported infinite beams with periodic and disordered repeating units. The disorders can be assumed to be present either due to beam elements not being equal in length or due to the presence of point disorders; which can be torsional springs or masses attached at a few or all supports.

A finite repeating unit may consist of (a) uniform beam of homogeneous material resting on unequally spaced rigid (simple) supports which divide the beam into N beam elements, (b) point springs attached at i supports and (c) point masses attached at j supports of the repeating unit, as shown in Figure 2.

The beam elements have lengths $l_1, l_2, l_3, \ldots, l_n$ between the supports (with total length l and average length $l_a = l/N$), and the point spring and point mass elements have torsional stiffness $K_1, K_2, K_3, \ldots, K_i$ and mass moments of inertia $I_1, I_2, I_3, \ldots, I_j$, respectively. The point elements are assumed to be placed at junctions of the ends of adjacent beam elements resting on common supports. Since the point elements, and also the beam elements, can have only rotational degrees of freedom at their ends coupled together at the supports, all of the elements are thus coupled through single co-ordinates. The theory developed in the preceding section is, therefore, applicable to the system considered here. The generalized displacements (q) and forces (F) will now have slopes (θ) and moments (M), respectively, at the simple supports. For continuity, the slopes of the beam elements resting on common supports, with or without a point element (or disorder), must be the same at their ends coupled together. For the condition of equilibrium of moments to be satisfied, the moments at the ends of two beam elements resting on a common support with point elements in between, will have to be different. The difference in moments will be equal to the torsional resistance offered by the point springs or masses at the supports.

The beam elements, being linear and uniform and hence symmetric, can be characterized by one direct and one transfer end receptance. The direct and transfer receptances of linear point elements are equal. The receptances for the beam, point spring and point mass elements are given in Appendix B. To present various results, a non-dimensional frequency parameter (Ω) is used. It is defined in terms of the average span length l_a as

$$\Omega = \omega l_a (m_b/EI)^{1/2}, \tag{5}$$

where ω is the angular frequency, and m_b is the mass per unit length of the beam of flexural rigidity EI. Other parameters used to present the results are the non-dimensional lengths $L_1 = l_1/l_a$, $L_2 = l_2/l_a, \ldots, L_n = l_n/l_a$, and the non-dimensional torsional stiffness k_r and mass moment of inertia I_m, defined in Appendix B.

Free wave propagation, as governed by the variation of the propagation constant μ_D with frequency, is studied for periodic beams with different amounts of point disorders. A simple two-span repeating beam unit without any point disorder has been considered to evaluate the effect of disorder arising form the variation of intermediate support location. The propagation zones and also the bounding and natural frequencies are studied as they vary with support location. The results are presented and discussed in the sections to follow.

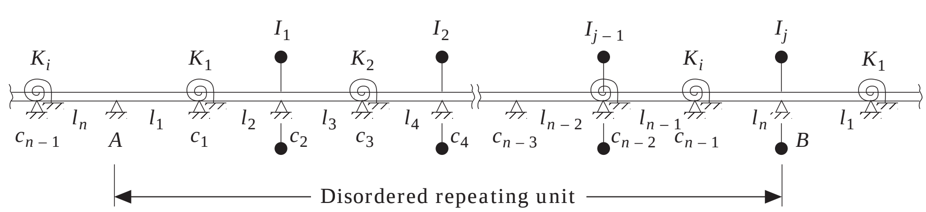


Figure 2. A disordered repeating beam on simple supports with torsional springs and rotary masses attached at some of the supports.

4. FREE WAVE MOTION IN PERIODIC BEAMS WITH POINT DISORDERS

To study the effect of point disorders on free wave motion in undamped infinite periodic beams, point springs or point masses are assumed to be mounted on the beam at alternate supports. Such an arrangement provides a simple model of repeating unit consisting of a two-span beam on simple supports with a point disorder mounted at the intermediate support located at the centre. In the absence of point disorder the system reduces to a periodic beam.

The variation of real and imaginary parts of the propagation constants have been studied for different amounts of point disorders (and without them) over a frequency range $\Omega = 0$–200. Results are presented, however, only for single values of $k_r = 4{\cdot}0$ and $I_m = 0{\cdot}1$ for the torsional spring and the mass, respectively, as shown in Figures 3(a) and (b) over a frequency range $\Omega = 0$–68, covering up to over two propagation zones of the periodic beam.

The value of the real part of the propagation constant, also known as the attenuation constant (μ_{Dr}) which is non-zero in the attenuation zones, corresponds to the attenuation that occurs across a two-span repeating beam unit. The magnitude of the imaginary part μ_{Di} of the propagation constant is either zero or π in the attenuation zones and it is defined as $0 \leqslant \mu_{Di} \leqslant \pi$ in the propagation zones. In the absence of any disorder, $\mu_{Dr} = 0$ in the primary propagation zones.

In the presence of point spring or point mass disorder, the primary propagation zones are altered and divided into two parts due to the appearance of intermediate attenuation zones. The repeating units considered being symmetric about their mid-points, the bounding frequencies coincide with the natural frequencies of the repeating unit [27, 28]. It may also be noted that all the attenuation zones are bounded by one SS frequency and one CC frequency, whereas the propagation zones can be bounded by both (i) SS frequencies, (ii) both CC frequencies or (iii) one SS frequency and one CC frequency; just like the repeating unit consisting of beam length disorders studied earlier [27]. (Conditions (iii) do not apply to the point spring disorder studied here.) The study of the variation of μ_{Dr} is useful to find which natural frequency will constitute an upper or lower bounding frequency of the PZs. The symmetric mode natural frequencies of the repeating beam unit with the extreme ends either simply supported ($\Omega = 15{\cdot}42$ and $49{\cdot}97$) or clamped ($\Omega = 22{\cdot}37$ and $61{\cdot}67$) are not affected due to the presence of point disorder at a mid-support, as indicated by the mode shapes shown corresponding to these frequencies in Figures 3(a) and (b). Thus, any change in the value of k_r or I_m will not change those natural frequencies which constitute one of the bounding frequencies of the intermediate propagation zones. The anti-symmetric mode natural frequencies, however, depend highly upon the type and amount of point disorder.

As k_r increases, coupling of the beam elements at the intermediate support weakens and μ_{Dr} attains higher values (Figure 3(a)) and the bounding frequencies that correspond to the anti-symmetric natural modes are raised. This results in narrowing down of the PZs and increases the modal density of a finite beam of periodically disordered elements. When k_r is very large ($k_r = \infty$), these frequencies become equal to the immediately higher symmetric mode frequencies. In such a case, the system is actually clamped at the alternate supports and free waves can exist only at these discrete frequencies.

In the presence of the point mass disorder the maximum value of μ_{Dr} in the attenuation zones increases with frequency and the propagation zones become narrow. The presence of a point mass in the repeating unit lowers its unsymmetric mode natural frequencies and hence the bounding frequencies which they identify. For high

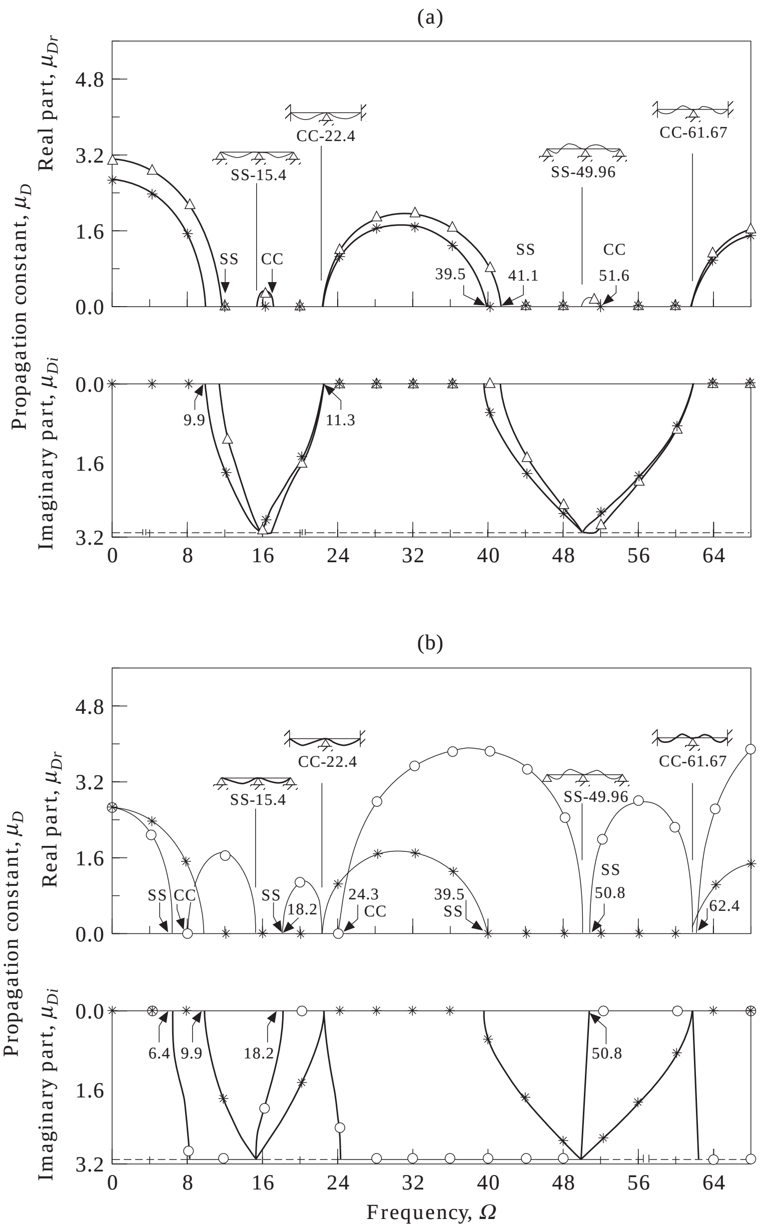


Figure 3. Variation of the real part μ_{Dr} and the imaginary part μ_{Di} of the propagation constants with frequency. (a) Periodic beam with and without a torsional spring disorder; (b) periodic beam with and without rotary mass disorder. —∗—∗—∗, Periodic beam; —△—△—, periodic beam with point spring disorder, $k_r = 4{\cdot}0$; —○—○—, periodic beam with point mass disorder, $I_m = 0{\cdot}1$.

values of I_m (as in this case, $I_m = 0{\cdot}1$), two intermediate attenuation zones have appeared in the first propagation zone. This is because the higher anti-symmetric mode natural frequency has decreased to a value ($\Omega = 18{\cdot}2$) which lies in the first propagation zone. Such a situation will arise in a higher primary propagation zone for a lower value of I_m.

5. RELATIONSHIP BETWEEN NATURAL AND BOUNDING FREQUENCIES

This section deals with the relationship between the natural frequencies of the repeating beam unit when disconnected from the rest of the system, and the bounding frequencies and bandwidths of free wave propagation zones. To gain an insight into the phenomenon that governs the free flexural wave motion in an infinite periodic beam having regularly disordered symmetric and unsymmetric repeating beam units, it is convenient to consider only a two-span repeating beam unit with a moveable intermediate simple support (prop).

The variation of fundamental and higher mode natural frequencies (and some mode shapes) of two-span beams with SS and CC ends has been studied separately [31] for prop locations defined by $0 \leqslant X \leqslant 2$, where $X = L_1$, the normalized length of element l_1. The length ratio $l_1/l_2 = L_1/L_2 = X/(2 - X)$. The variation with X of the bounding frequencies of the propagation zones is presented in Figure 4. The bandwidths of various propagation

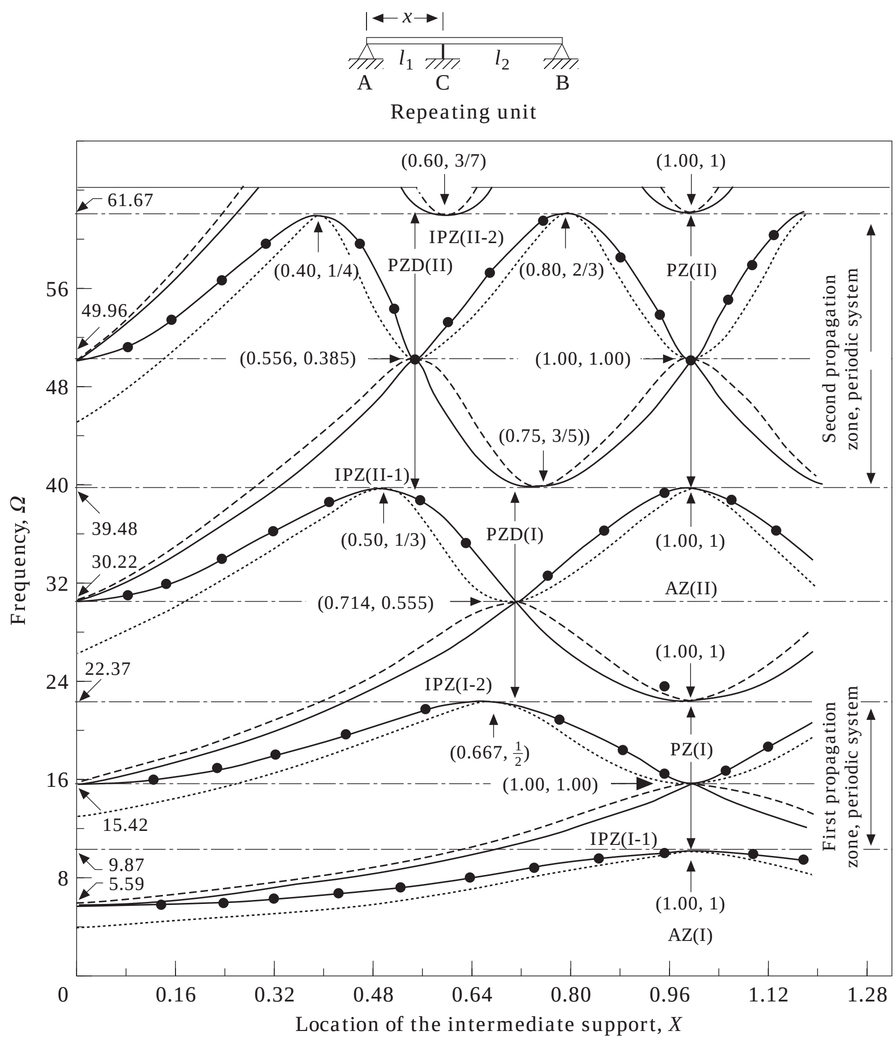


Figure 4. Variation of the natural and bounding frequencies of free wave propagation with location X of the intermediate support of a two span repeating beam unit. —●—●—, Lower bounding frequencies; —, higher bounding frequencies; · · · ·, resonance frequencies with extreme ends simply supported and -----, resonance frequencies with extreme ends clamped.

and attenuation zones as they vary with X may be read off the frequency axis. The SS and CC natural frequencies of the repeating beam unit over a frequency range $\Omega = 0$–64 fully covers up to the second mode of the periodic beam ($X = 1{\cdot}0$). The frequency curves for the prop locations $1 \leqslant X \leqslant 2$ being the mirror image of the curves for $0 \leqslant X \leqslant 1$, it is sufficient to consider the frequency curves over the range $X = 0$–1, although these are shown up to $X = 1{\cdot}2$.

When the beam segments are unequal ($X \neq 1$) an intermediate attenuation zone (IAZ) will appear in each of the primary propagation zones (PPZs) and divide them into two IPZs in general. These are marked as IPZ (I-1), (I-2), (II-1) and (II-2) in Figure 4. I and II indicate number of the PPZ to which the IPZs 1 and 2 belong. As the props are moved towards the end supports, the IPZs narrow down. For $X = 0$, the infinite beam will rest on very closely placed twin simple (hinged) supports so that the repeating beam becomes effectively clamped at its ends with clear span of length $2l_a$ and free waves can now exist only at discrete frequencies ($\Omega = 5{\cdot}60$, 15·54, 30.22 and 49.97) which are the CC frequencies of the beam without the intermediate support. For a prop location $0 < X < 1$, the repeating units of the regularly disordered infinite periodic beam are unsymmetric. For such systems, the SS and CC frequencies normally lie inside the attenuation zones [26, 28], except for some specific prop locations when the natural frequencies coincide with the bounding frequencies; just as in the case of a repeating unit that is symmetric ($X = 1$). This occurs when the natural and bounding frequency curves together attain some maximum (or minimum) values and coincide. The specific prop locations and length ratios (X, l_1/l_2) for which the natural and the bounding frequencies coincide are shown in Figure 4 together with the IPZs.

For specific values of (X, Ω) at which the natural and bounding frequencies coincide, the end responses of the unsymmetrically disordered beams (just as in the case of a symmetric beam, $X = 1$) must be equal in magnitude, but can be either in phase or in counterphase [27, 31].

5.1. A NATURAL FREEQUENCY COINCIDING WITH A BOUNDING FREQUENCY

When only one of the SS and CC frequencies coincides with a bounding frequency of an IPZ, the beam segments on the two sides of the intermediate supports(s) vibrate in such modes that their end conditions are identical with those of the extreme ends of the repeating beam, either both simply supported or both clamped [31]. The ends of the segments resting at the common intermediate supports assume the same end conditions as those of the other (outer) ends. The two segments can, however, vibrate in different modes depending upon the prop location.

For a periodic beam ($X = 1{\cdot}0$) the resonance frequencies ($\Omega = 9{\cdot}87$, 22·37, 39·48 and 61·67) coincide with the bounding frequencies of the propagation zones. For the periodically disordered beam ($0 < X < 1$), a situation similar to this arises at $\Omega = 22{\cdot}37$ ($X = 0{\cdot}667$), 39·48 ($X = 0{\cdot}5$ and 0·75) and 61·67 ($X = 0{\cdot}4$, 0·6 and 0·8). From the knowledge of the mode shapes of the beam system [27, 31], it is easy to see that the locations of intermediate support(s) for which the natural frequencies coincide with the bounding frequencies of the propagation zones are the nodal points of the repeating beam unit. For the periodic beam, $X = 1{\cdot}0$ is also one of the nodal points of the same beam. Thus the resonance frequency(ies) of an unsymmetric repeating beam unit can lie at the bound(s) of the propagation zone(s), provided that the intermediate support(s) is(are) located at the nodal point(s) of the beam vibrating in flexure and the end conditions assumed by the segments resting at the common intermediate support(s) are similar to the identical extreme end conditions of the repeating beam unit.

5.2. TWO RESONANCE FREQUENCIES COINCIDING WITH TWO BOUNDING FREQUENCIES

For an N-span periodic beam repeating unit, $N-1$ of its SS and CC frequencies coincide (occur in pairs) and lie inside the propagation zones [26, 28]. When the two spans of the beam considered are equal ($X = 1$, periodic) one each of the SS and CC frequencies and the two bounding frequencies (and also the IPZs) join together at $\Omega = 15{\cdot}42$ and 49·96. Situations similar to these occur at $X = 0{\cdot}556$, when the repeating beam unit is unsymmetrically disordered. For such a beam, when one each of the SS and CC frequencies together coincide with the two bounding frequencies of the propagation zones (where two IPZs belonging to the same or different PPZs merge together), the beam segments on the two sides of the intermediate support(s) vibrate in such modes that their ends resting on the common intermediate support(s) assume end conditions that are different from those at their other (outer) ends. From the knowledge of mode shapes [31] it can easily be seen that the prop location(s) for which SS and CC frequencies (and also two bounding frequencies) coincide are also the nodal points of the beams vibrating in flexure.

The values of (X, Ω) for which SS and CC frequencies occur in pairs and lie inside a propagation zone are (0·714, 30·22) and (0·556, 49·96). For these values of (X, Ω) IPZs join together (and the intermediate AZs vanish altogether), just as in the case of periodic repeating beam unit ($X = 1$). At $X = 0{\cdot}7142$, IPZ (I-2) and IPZ (II-1) join together and at $X = 0{\cdot}556$ IPZ (II-1) and IPZ (II-2) join together to provide propagation zones PZD(I) and PZD(II) of the disordered periodic beam. The real parts of the propagation constants $\mathrm{Re}(\mu_D)$ of two-span disordred and periodic repeating beams over a frequency range $\Omega = 0$–80 are compared in Figure 5. The AZ and PZ for the periodic beam are marked on the figure. The propagation zone PZD(I) for the disordered unsymmetric system ($X = 0{\cdot}7142$), in which the coinciding SS and CC natural frequencies ($\Omega = 30{\cdot}22$) lie, is broader than the corresponding propagation zone PZ(I) of the periodic beam. Likewise, the PZD(II) for $X = 0{\cdot}556$ is also slightly broader than the corresponding PZ(II) of the periodic system.

It is also interesting to note that the stop bands of some of the disordered systems are the pass bands of the periodic system. This is explained in the section that follows.

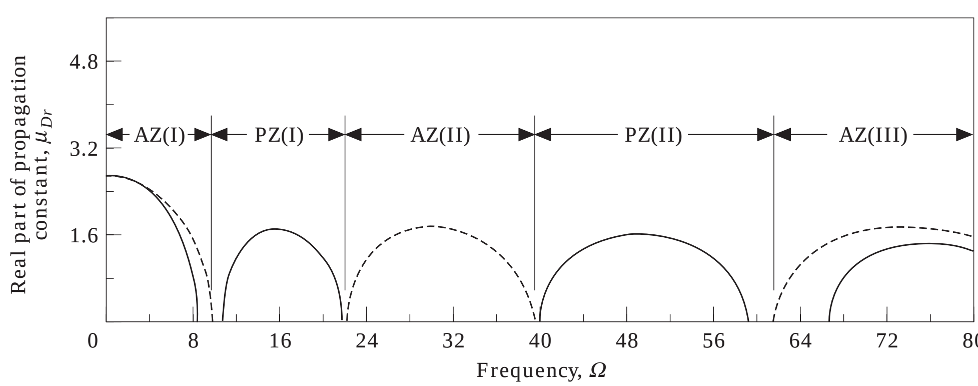


Figure 5. Variation of the real part of μ_{Dr} of the propagation constant with frequency. —, Periodic beam with a two unequal span ($L_1 : L_2 = 0{\cdot}714 : 1{\cdot}286$) repeating beam unit; - - - - -, periodic beam with a two identical span ($L_1 : L_2 = 1 : 1$) repeating beam unit.

6. DISORDERS FOR VIBRATION ISOLATION AND CONTROL OF WAVE PROPAGATION

The location of the intermediate support(s) in a repeating unit significantly influences the free wave motion in the periodic system. In the case of the two-span repeating beam unit considered here, the propagation zones are quite narrow when the intermediate support is close to either of the two end supports. The bandwidths of the propagation zones (and hence those of the attenuation zones) vary considerably as the disorder is varied by moving the intermediate support to mid-span ($X = 1{\cdot}0$), when the system becomes periodic. The frequency bands over which free waves can either propagate or attenuate in a periodic system are well defined. If the excitation frequencies (or frequency band) of a source of disturbance lie outside an attenuation zone, the waves/vibrations will propagate without attenuation when the system is undamped.

As discussed earlier, variation of disorder with the location of the intermediate support can altogether modify and even interchange the propagation and attenuation bands (see Figure 4). For instance, PZII ($\Omega = 39{\cdot}48$–$61{\cdot}67$) of the periodic system ($X = 1$) is almost completely replaced by an attenuation zone when the intermediate support is moved to a location between $X = 0{\cdot}75$ and $X = 0{\cdot}80$ ($l_1/l_2 = 3/5$ to $2/3$). One can thus design an appropriately disordered system to control free wave motion and hence minimize transmission of vibration. According to the frequency of excitation, a judicious choice of the disorder (the support location(s) in this case) is essential, as a wrong choice of support location(s) can introduce a propagation zone, which can be even broader than that of the corresponding zone of the periodic system.

The study and knowledge of propagation zones of the disordered periodic systems are therefore very essential and useful in designing new structures, where the propagation of waves and hence transmission of vibration over a frequency range need to be controlled. An appropriate location of intermediate support(s) can be selected from a graph such as the one given in Figure 4, such that the predominant frequency (or frequency range) of disturbance lies well inside an attenuation zone. Conclusions similar to those drawn in this study are also valid in the frequency range covering higher natural modes.

For the sake of convenience, only two-span repeating beam units have been considered in this work, to gain an insight into the special phenomenon that governs free flexural wave motion in regularly disordered periodic systems with unsymmetric repeating beam units. Beam systems having more than two-span repeating units can also be studied in a similar manner.

7. LOCALIZATION OF NORMAL MODES OF UNSYMMETRICALLY DISORDERED SYSTEMS

A finite beam resting on unequally spaced simple supports is said to be symmetric when its stiffness, mass and the constraints offered by the supports are all symmetric about its mid-point. While the finite system consisting of identical symmetric elements will always be symmetric, a system consisting of identical unsymmetric elements will be unsymmetric in general, but an even number of unsymmetric identical elements can be arranged to have a symmetric system. A randomly disordered system is always expected to be unsymmetric, as all of its elements are different. According to Hodges [15], normal mode localization leads to spatial decay in the vibration amplitude even at frequencies which are within the pass bands (PZ) of the tuned (periodic) system. This decay is known to be exponential, and the associated exponential decay constant is called the localization factor. The degree of localization is known to increase with the ratio of disorder strength to modal coupling. The localization, which in fact relates to the confinement of normal modes in the finite disordered system, can also be understood by considering the finite system to have cyclic end conditions. That is, the disordered system is considered to be the repeating unit of a periodically disordered infinite system. (Cyclic end conditions are closer to the actual end conditions encountered in real structures of a sufficiently large number of bays as compared to the standard end conditons (SS and CC ends) normally assumed in theoretical analysis.)

It is well known that in a periodic system the free waves progress and carry energy in frequency propagation zones, and that they attenuate (or decay) exponentially and do not carry energy in attenuation zones. $N-1$ out of the N natural frequencies of the N-element periodic repeating unit always lie inside the propagation zone and one each of the remaining frequencies of the system with SS and CC ends constitute the lower and higher bounding frequencies of the propagation zones, respectively. When the individual elements of the periodic repeating units are unsymmetric, the lowest and the highest natural frequencies of the finite system with SS and CC ends will lie inside the lower and the higher attenuation zones, respectively [28]. The bandwidth of PZ will thus be smaller than the frequency separation of the lowest and the highest frequency of the periodic system [28]. With the increase in number of periodic elements N in a finite system the modal density (the number of normal modes per unit frequency bandwidth) increases, although the bandwidths of the PZs remain unchanged. In the PZs, $\mu_r = 0$ and the bandwidths of the PZs and AZs of the periodic system are not affected by the number of identical elements considered in the repeating unit, whereas the value of the attenuation constant μ_{pr} is N times that of μ_r across a single element.

In what follows, it is argued that the phenomenon of wave confinement (localization) can be very conveniently interpreted and understood in terms of the attenuation constant μ_r.

7.1. EFFECT OF COUPLING AND BEAM LENGTH DEVIATION ON THE MODAL DENSITY OF FINITE SYSTEMS

As discussed in section 4, the increase in torsional stiffness k_r at the intermediate supports will reduce the coupling between the adjacent spans, increase μ_r and narrow down the PZs. The increase in the torsional stiffness of the spring increases the modal density. For very large values of k_r, the beam gets clamped at the supports, and the PZs are narrowed down to such an extent that the free waves that can exist only at the CC natural frequencies will be the standing waves. As the spans are equal in case of an ideally periodic beam, the normal modes will be collective (all the spans will have the same peak deflection) irrespective of the amount of coupling.

When the beam elements of the finite system are unequal, the primary PZs are divided into N IPZs (= the number of spans) due to the appearance of $N-1$ IAZs, and the effective width of free wave propagation is reduced; although the primary PZs of the disordered system can become broader than the corresponding PZs of the periodic system. As the number of beam elements and their length deviations from the perfect periodicity increase, the bandwidths of the IAZs (and the value of μ_r in them) increase, while the IPZs narrow down further. When the number of elements and/or length deviations are very large, the effective bandwidth of free wave propagation reduces considerably [27]. Thus, the modal density increases with a decrease in coupling and an increase in the number of beam elements and their length deviations.

7.2. SYMMETRIC AND UNSYMMETRIC SYSTEMS WITH LARGE DELIBERATE DISORDERS

On the basis of the study of the attenuation constant of periodically disordered systems and the deflection mode shapes of symmetric and unsymmetric finite repeating units, it can be understood how localization can occur in a finite disordered system [27, 31]. When the finite systems are disordered symmetrically, all of the bounding frequencies of its IPZs are identified with its SS and/or CC natural frequencies. Free waves corresponding to the natural frequencies are thus also the standing waves, which do not decay spatially as $\mu_r = 0$ at these frequencies. Just like the periodic systems, the symmetrically disordered and fully coupled systems are thus not expected to have their normal modes localized; although the peak deflections of excessively different span lengths can differ considerably to give the appearance of localized modes. When the finite systems are unsymmetrically disordered, their normal mode frequencies lie inside the attenuation zones. (Randomly disordered systems are inherently unsymmetric, as all of their elements are expected to be different.) The number and the widths of the IAZs, and also the value of the attenuation constant μ_r (in the IAZs), increase with an increase in the number of beam elements and their length deviations [27]. Thus the modal density can be considerably high even if the system is strongly coupled. Localization can therefore occur in strongly coupled unsymmetric systems provided that they are strongly disordered [27, 31].

As discussed in section 5, the beam-type system, which is highly disordered due to intermediate support(s) which happen to occupy the nodal points of the beam vibrating in higher modes, can behave like an ideally periodic beam with its modes fully de-localized or collective. If the mistuning is low, modes can be localized only if the number of randomly disordered elements is very large or the coupling is weak.

7.3. THE EFFECT OF DAMPING

Localization in the presence of damping is due to dissipation of energy as the waves progress. It increases with frequency and it is equally effective in both symmetrically and unsymmetrically disordered systems and also in periodic systems [26]. The presence of damping renders the attenuation constant μ_r non-zero in the propagation zones, and it increases with the amount of damping and the frequency. The effect of damping is more pronounced in the primary PZs of the disordered system than in that of the periodic system. The effect of damping on wave confinement in periodically disordered systems is thus similar to that of localization.

7.4. PRIMARY PROPAGATION ZONES OF DISORDERED SYSTEMS

As indicated earlier in section 1, the PZs and AZs of a periodic system become PPZs and PAZs, respectively, when the finite repeating system is disordered. PPZs are broader than PZs, and PAZs are narrower than AZs. Intermediate propagation and attenuation zones (IPZs and IAZs) belong to and lie in the PPZs only. Randomly disordered finite systems have, in general, all of their elements unequal and are thus unsymmetric. They will therefore normally have the frequencies of their localized normal modes lie inside the IAZs and PAZs of the periodically disordered system [26, 27]. Thus some of the localized-mode frequencies of the randomly (unsymmetrically) disordered systems that lie inside the IAZs may be said to lie indirectly in the PPZs, as the IAZs further lie within the PPZs. Some of such frequencies lie in the pass bands (PZs) that correspond to those of the ideally periodic systems, as is already known [19].

The values of the μ_r, as apparent from the IAZs that appear in the primary PZs of periodically disordered systems consisting of four, eight beam elements [26, 27], vary from pass band to pass band, and also across the pass bands. The IAZs gradually grow in size as we move from the lowest (first) to the highest (sixth) pass band. Even a fully coupled disordered system, which may have only weakly localized modes in the first pass band, can become strongly localized in a sufficiently higher pass band where μ_r attains higher values. This is also exactly the same as expected according to the literature on normal mode localization [19]. The degree of localization is known to be a minimum [19, 21] at the mid-pass bands, and it increases as one moves towards its edges. This, however, holds good only for the first three pass bands of the system studied earlier [27]. As the IAZs at the mid-pass bands grow at a faster rate from pass band to pass band, μ_r, and hence the localization at the middle of the higher pass bands, attains higher values than those across the rest of the pass bands. Thus, the information available on normal mode localization does not appear to hold good for the higher pass bands—this needs further investigation. IAZs appear in the primary PZs and attain significant values only if the system is disordered strongly enough. A weakly disordered system, if not weakly coupled, will be localized only if the number of system bays and/or the frequency band is very high.

8. CONCLUSIONS

The general theory developed for free wave motion in infinite mono-coupled periodic systems with multiple disorders has been found to be useful in studying the free flexural wave motion in periodic beams with beam length disorders, point mass disorders and point spring disorders.

In the presence of point (or rotary) disorders, the primary propagation zones are divided, in general, into as many intermediate propagation zones as the number of beam elements in the disordered repeating unit. The number of intermediate propagation zones and attenuation zones is not further affected when spring disorders of finite torsional stiffness are present at the intermediate supports. In the presence of rotary mass disorders, the number of intermediate attenuation zones can be as many as the number of intermediate propagation zones in some particular primary propagation zone. Rotary mass disorders cause very large attenuation and further narrow down the propagation zones in the higher frequency range. These can therefore be used to isolate high frequency disturbing sources. The presence of rotary mass disorder, just like the beam length disorder, always lowers the lower bounding frequencies of the primary propagation zones, whereas the presence of torsional spring disorder always raises these frequencies. An increase in the stiffness of the point spring weakens the coupling between the beam elements and narrows down the propagation zones, thereby increasing the modal density.

The bounding frequencies of propagation zones for symmetric beams with point disorders are identified with the natural frequencies of the repeating unit with its extreme ends simply supported (SS ends) or clampled (CC ends). Just like the system with beam length disorders, the propagation zones of symmetric beam systems with point mass disorder may be bounded by both SS frequencies, both CC frequencies, or one SS frequency and the other CC frequency, whereas the attenuation zones are always bounded by one SS frequency and one CC frequency. It is not possible to know in advance which natural frequency will identify with a lower or higher bounding frequency of propagation and attenuation zone. The bounding frequencies of the intermediate propagation zones cannot in general be identified with the natural frequencies of the unsymmetrically disordered repeating beam unit. The natural frequencies normally lie in the attenuation zones, but under certain conditions they can lie at the boundaries of the propagation zones and even inside them. This happens when the intermediate support(s) is (are) located at the nodal point(s) of the beam vibrating in flexure.

This study has revealed that all of the primary propagation zones are not necessarily divided into as many intermediate propagation zones as the number of beam elements of repeating units with beam length disorders. When the intermediate support(s) occupy position(s) that correspond(s) to the nodal point(s) of the repeating beam unit, intermediate propagation zones can join together just like the intermediate propagation zones of a periodic system having multi-span periodic repeating units. The resulting propagation zone can be broader than the primary propagation zone. This occurs when the SS and CC frequencies of the disordered beam coincide. The introduction of appropriate amount(s) of disorder in a periodic system can help to replace pass band(s) with some stop band(s).

Since the frequency bands of free wave propagation depend upon the location of the intermediate support(s) of the repeating unit, the present study should be helpful in designing new structures in which the transmission of free waves and vibration are required to be controlled. A graph for the variation of attenuation zones with disorder, like the one developed in this paper, can be used to tackle specific problems.

Confinement (attenuation) of free waves corresponding to the normal modes of unsymmetrically disordered beam-type finite systems has been discussed in terms of the attenuation constant of periodically disordered systems, and it has been pointed out that it is very useful in a qualitative study of the phenomenon of normal mode localization. The trend in variation of wave confinement with coupling, disorder, damping and frequency, based on variation of the attenuation constant, has been found to be similar to the information available in the literature on normal mode localization, except that the attenuation constant and hence the wave confinement (localization) at mid-pass bands increases more rapidly from pass band to pass band. The localization, which is known to be a maximum at the mid-pass bands, does not appear to hold good, in general, for higher pass bands and should be studied further.

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APPENDIX A: END RECEPTANCES OF AN ARRAY OF MONO-COUPLED ELEMENTS

The direct end receptances α_{CAA} and α_{CBB} and cross end receptances β_{CAB} and β_{CBA} of an array of N mono-coupled elements, as considered in this paper, can easily be found [30] in terms of the receptances $\alpha_{lln'}$, α_{rrn}, β_{lrn} and $\beta_{rln}(n = 1, 2, 3, \ldots, N)$ of the individual elements. These are listed here:

$$\alpha_{CAA} = \alpha_{ll1} - \beta_{lr1}\beta_{rl1}|A_{11}|/|A|, \qquad \alpha_{CBB} = \alpha_{rrN} - \beta_{lrN}\beta_{rlN}|A_{N-1,N-1}|/|A|, \quad \text{(A1, A2)}$$

$$\alpha_{CAB} = (\beta_{lr1}\ldots\beta_{lr2}\ldots,\beta_{lrn})/|A|, \qquad \alpha_{CBA} = (\beta_{rl1},\beta_{rl2},\ldots,\beta_{rln})/|A|. \quad \text{(A3, A4)}$$

Here $|A|$, $|A_{11}|$ and $|A_{N-1,N-1}|$ are the determinants of $[A]$ and of its cofactors of elements a_{11} and $a_{N-1,N-1}$, respectively. $[A]$ is a tri-band square matrix of order $N-1$. Its principal diagonal has the elements $a_{ii} = \alpha_{rri} + \alpha_{ll(i+1)}(i = 1, 2, \ldots, N-1)$ and the elements of the other two diagonals are $a_{i,i+1} = -\beta_{lr(i+1)}$ and $a_{i+1,i} = -\beta_{rl(i+1)}$ $(i = 1, 2, \ldots, N-2)$. The rest of the elements of the matrix $[A]$ are zero.

The frequencies at which the above receptances attain infinite values are the resonance frequencies of the N-element finite system with both of its ends free. The direct receptances α_{CAA} and α_{CBB} can be modified to α^c_{CAA} and α^c_{CBB} [30] for the ends B and A fixed, respectively.

This is achieved by modifying the matrices $[A]$, $[A_{11}]$ and $[A_{N-1,N-1}]$. The expression for α^c_{CAA} is obtained from equation (A1) by modifying $[A]$ by adding the quantity $(-\beta_{rlN}/\alpha_{rrN})$ to its element $a_{N-1,N-1}$. Likewise α^c_{CBB} is obtained from equation (A2) by modifying $[A]$ by adding the quantity $(\beta_{lr1}/\alpha_{ll1})$ to its elements a_{11}. The determinants of $[A]$ and also that of its co-factors $[A_{11}]$ and $[A_{N-1,N-1}]$ are also modified accordingly. The frequencies at which α^c_{CAA} and α^c_{CBB} are large are the natural frequencies of the finite system with the right and the left end fixed, respectively. The frequencies at which these are zero are the natural frequencies of the finite system with both ends fixed.

APPENDIX B: RECEPTANCES OF DIFFERENT TYPES OF ELEMENTS

B.1. BEAM ELEMENT

The receptances of the beam elements are given in reference [32] and are reproduced here. For a symmetric beam element, the two direct end receptances are equal and are given by

$$\alpha_{ll} = \alpha_{rr} = \alpha = (\coth k_s l - \cot k_s l)/(2EIk_s). \tag{B1}$$

Also, for a linear beam element, the two transfer end receptances are equal and are given by

$$\beta_{lr} = \beta_{rl} = \beta = (\operatorname{cosech} k_s l - \operatorname{cosec} k_s l)/(2EIk_s). \tag{B2}$$

where EI is the flexural rigidity of the beam element of length l. k_s is the structural wavenumber, defined by

$$k_s = \omega^{1/2}(m_b/EI)^{1/4}, \tag{B3}$$

where m_b is the mass per unit length of the beam element and ω is the angular frequency (rad/s).

B.2. POINT MASS ELEMENT

The torsional receptance of the point mass having moment of inertia I is given by α (mass) $= -1/I\omega^2$ which, when expressed in terms of non-dimensional quantities, becomes

$$\alpha(\text{mass}) = -(l/\Omega^2)/(EII_m), \tag{B4}$$

where $I_m = I/(m_b l^3)$ is the non-dimensional mass moment of inertia of the point mass. Ω has been defined in equation (5) for a beam element of length l_a.

B.3. POINT SPRING ELEMENT

The receptance of a point torsional spring with a torsional stiffness K is given by $\alpha(\text{spring}) = 1/K$ which, when expressed in terms of the non-dimensional spring stiffness $k_r = Kl/EI$, becomes

$$\alpha(\text{spring}) = l/(EIk_r). \tag{B5}$$